Few-Body Syst (2011) 49: 177–181
DOI 10.1007/s00601-010-0099-3

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Light-Front Singularities

Received: 5 July 2010 / Accepted: 31 July 2010 / Published online: 22 August 2010

Abstract Calculations in light-front quantization are sometimes found to lead to singularities that are not present in the corresponding manifestly covariant treatment. We give some examples that were found in the framework of perturbation theory, but must also occur in nonperturbative calculations. In the case of anomalies, regularization-scheme dependences were found that not only occur between the light-front approach and manifestly covariant calculations, but also among the latter ones.

1 Introduction

Light-Front Dynamics (LFD) is ideally suited for a description of relativistic processes because:

(i) A Fock-space expansion of many-particle states is valid owing to the simplicity of the Fock vacuum. This property is connected to the peculiar dispersion relation of light-cone momentum and energy

$$k^{\pm} = \frac{k^0 \pm k^3}{\sqrt{2}}, \quad \boldsymbol{k}_\perp = (k^1, k^2), \quad k^- = \frac{m^2 + \boldsymbol{k}_\perp^2}{2k^+}, \quad k^+ \geq 0. \tag{1}$$

(ii) In LFD one works with physical degrees of freedom only. No negative-energy particles are included and the LF gauge is free of ghosts.

(iii) LFD treats physical systems at the amplitude level: LF wave functions are defined independently of the reference frame, i.e., they are boost invariant.

Generally speaking, there are two approaches to LFD: Project covariant amplitudes on the light front by integrating over the light-front energy k^- [1], or determine the light-front Hamiltonian and quantize it [2]. In both approaches one may encounter numerous treacherous points. Presently our taxonomy is as follows:

Type I singularities $\int dk^-$ does not converge.

Type II singularities The covariant amplitude A_{cov} is finite, but the LF amplitude A_{LF} diverges.

Anomalies The amplitude is divergent and after renormalization $A_{\text{cov}} \neq A_{\text{LF}}$.

Interestingly, as we shall show in Sect. 4, we found that there are even differences between the results of the manifestly-covariant calculations of the one-loop correction to the γ-vector–boson coupling. Fortunately, these differences cancel out in the Weinberg–Salam sector of the Standard Model owing to a symmetry.

Contribution to the workshop “Relativistic Description of Two and Three-Body Systems in Nuclear Physics”, ETC*, October 19–23, 2009.

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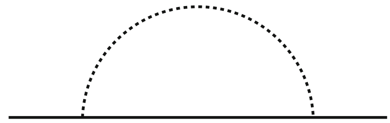

Fig. 1 Self-energy diagram

2 Type I LF Singularities

Type I singularities were seen before by several authors, but given different names, if any: Pauli and Brodsky [3] *self-induced inertias*, Burkardt [4] *finite mass renormalization*, Schoonderwoerd and Bakker [5] *forced instantaneous loops*, Bakker and Ji [6], Bakker et al. [7], Misra and Warawdekar [8], Bakker et al. [9] *arc and point singularities.*

As one out of numerous examples we discuss the vector two-point function in (1+1) dimensions [9] (Fig. 1). The amplitude is given by

$$V^{\mu} = \int d^2k \frac{k^{\mu}}{D_1 D_2}, \quad D_1 = k^2 - m_1^2 + i\epsilon, \quad D_2 = (k-p)^2 - m_2^2 + i\epsilon \tag{2}$$

with the manifestly covariant result $V^{\mu} = p^{\mu} I_{\text{cov}}$ and

$$I_{\text{cov}} = -i\pi \int_0^1 dx \frac{x}{D_{\text{cov}}}, \quad D_{\text{cov}} = x(1-x)p^2 - (1-x)m_1^2 - xm_2^2. \tag{3}$$

Here, x is the Feynman parameter.

To evaluate the same integral in the light-front we perform the integration in the complex k^- plane. $(k^2 - 2k^+k^-)$

$$V^{\mu} = \int dk^+ dk^- \frac{k^{\mu}}{D_1 D_2}, \tag{4}$$

Carrying out the integration over k^- first, one notices that the integral over this variable converges only for the plus component ($\mu = +$). In that case there are two poles in the complex k^--plane, which lie in different half planes only for $0 < k^+ < p^+$. Using the residue theorem and making the transformation $k^+ = xp^+$, one finds that the result agrees fully with the covariant result.

For the minus component ($\mu = -$), the integration is a bit more tricky. The naive residue calculus gives the result

$$I_{\text{LF}^-}^{\text{residue}} = -\frac{i\pi}{p^2} \int_0^1 dx \frac{(1-x)p^2 + m_1^2 - m_2^2}{D_{\text{cov}}} + \frac{i\pi}{p^2} \int_0^1 \frac{dx}{x}. \tag{5}$$

This result is incorrect because there is an infinity due to $\int_0^1 dx/x$, and the finite part differs from the covariant result. The root of this problem is the fact that the integral over k^- *diverges*. As the integral over k defining the amplitude Eq. (2) is superficially convergent, and the application of the residue theorem gives an erroneous result due to the fact that the contour at infinity gives a divergent contribution, we call this singularity an *arc* singularity.

If one takes the scalar product of V^{μ} with p^{μ} one finds that I is given by

$$I = \frac{1}{2p^2} \int d^2k \left(\frac{1}{D_2} - \frac{1}{D_1} + \frac{p^2 + m_1^2 - m_2^2}{D_1 D_2} \right). \tag{6}$$

The last integral can be evaluated in a straightforward way integrating over k^- first and using the residue theorem

$$\int d^2k \frac{1}{D_1 D_2} = -i\pi \int_0^1 dx \frac{1}{D_{\text{cov}}}. \tag{7}$$

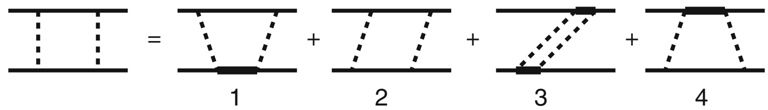

Fig. 2 LF amplitudes corresponding to the covariant box

The two integrals with a single denominator, *tadpoles*, can be treated using a method pioneered by Ligterink and Bakker [10]. In the latter work a method invented by Yan [11] was utilized. See also [12]. The tadpoles, which are necessarily singular because of the arc contributions, are in Ref. [10] found to be

$$A_0 = \int d^2k \frac{1}{k^2 - m^2 + i\epsilon} = \lim_{\Lambda\to\infty,\delta\to 0} -i\pi[\log(\Lambda/i\delta) - \log m^2]. \tag{8}$$

This object would vanish identically if one would apply the residue theorem in a naive way to the LF calculation, but dimensional regularization gives a result with the same dependence on the mass.

Using these results one finds

$$I = -\frac{i\pi}{2p^2}\left[\int_0^1 dx \frac{p^2 + m_1^2 - m_2^2}{D_{\text{cov}}} + \log\left(\frac{m_1^2}{m_2^2}\right)\right], \tag{9}$$

which is easily found to be identical to the covariant result.

3 Type II LF Singularities

Type II singularities correspond to finite integrals over k^-. The residue calculus is correct; no arc contributions occur. As an example consider the box diagram in the Yukawa model, spin-1/2 'constituents' and spin-0 'exchanged' particles, which is finite [13] (Fig. 2).

The covariant box is given by $\mathcal{T} = \bar{u}(p_1', s_1')\bar{u}(p_2', s_2')\mathcal{M}u(p_1, s_1)u(p_2, s_2)$; the invariant amplitude is then (k_i are linear combinations of k and p_i and p_i')

$$\mathcal{M} = \int \frac{d^4k}{(2\pi)^4} \frac{(\gamma(1)\cdot k_2 + m)\;(\gamma(2)\cdot k_4 + m)}{(k_1^2 - \mu^2)(k_2^2 - m^2)(k_3^2 - \mu^2)(k_4^2 - m^2)}. \tag{10}$$

The usual k^--integration to expand the covariant box in light-front amplitudes gives four light-front amplitudes

$$\mathcal{T} = \sum_d \mathcal{T}^d, \quad d = 1, \ldots 4.$$

We use the *blink* construction here to remove the cancelling singularities of the fermion propagators, see Ref. [10]. The LF amplitude with blinks is obtained adding the amplitude with an instantaneous fermion propagator to the amplitude with LF fermion propagators.

It turns out that all LF time ordered boxes are divergent. These divergences are due to the k^- dependence of the numerators of the fermion propagators. Therefore we need to regularize the LF amplitudes. We used two different schemes: DR_2, dimensional regularization in the perpendicular momenta, where the singular parts are proportional to $1/(D-2) = 1/\bar{\epsilon}$ and Pauli–Villars for the boson where the singular part is proportional to $\log(\Lambda^2)$. The result is [13] that the sum of all divergences vanishes:

$$\text{DR}_2: \quad C_d\,\frac{1}{\bar{\epsilon}}, \qquad \text{Pauli}-\text{Villars}: \quad C_d\,\log\Lambda^2, \quad \sum_{d=1}^{4} C_d = 0. \tag{11}$$

Clearly, it is important to retain the *stretched box* $\mathcal{T}_3$, otherwise the singularities do not cancel.

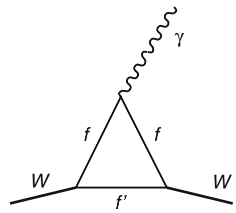

Fig. 3 The lowest-order correction to the γWW vertex

4 Light-Front Anomalies

In order to see if in fundamental theories surprises might occur, we studied the Weinberg–Salam sector of the Standard Model, in particular the renormalization of the CP-even EM weak boson vertex [14]. The Lorentz-covariant and gauge-invariant CP-even electromagnetic γW^+W^--vertex is defined in the literature:

$$\Gamma^\mu_{\alpha\beta} = ie\Bigg\{A[(p+p')^\mu g_{\alpha\beta} + 2(g^\mu_\alpha q_\beta - g^\mu_\beta q_\alpha)] \\ +(\Delta\kappa)(g^\mu_\alpha q_\beta - g^\mu_\beta q_\alpha) + \frac{\Delta Q}{2M_W^2}(p+p')^\mu q_\alpha q_\beta\Bigg\}, \tag{12}$$

At tree level, $A = 1$, $\Delta\kappa = 0$, $\Delta Q = 0$, for any $Q^2 = -q^2$ because of the point-like nature of $W^\pm$ gauge bosons, but beyond tree level

$$A = F_1(Q^2), \quad -\Delta\kappa = F_2(Q^2) + 2F_1(Q^2), \quad -\Delta Q = F_3(Q^2), \tag{13}$$

where F_1, F_2 and F_3 are defined by the relation to the current matrix elements: i.e., $\Gamma^\mu_{\alpha\beta} = -ieJ^\mu_{\alpha\beta}$ and

$$J^\mu_{\alpha\beta} = \Bigg\{-(p+p')^\mu g_{\alpha\beta}\ F_1(Q^2) + (g^\mu_\alpha\, q_\beta - g^\mu_\beta\, q_\alpha)\ F_2(Q^2) + \frac{q_\alpha q_\beta}{2M_W^2}(p+p')^\mu F_3(Q^2)\Bigg\}.$$

In the light-front approach we cannot utilize this manifestly covariant formulation, but rather rely on the matrix elements $G^\mu_{h'h}$ in the space of the spin wave functions of the W bosons (Fig. 3):

$$G^\mu_{h'h} = -\epsilon^*_{h'}\cdot\epsilon_h(p+p')^\mu\ F_1(Q^2) + (\epsilon^\mu_h\ q\cdot\epsilon^*_{h'} - \epsilon^{*\mu}_{h'}\ q\cdot\epsilon_h)\ F_2(Q^2) \\ + \frac{(\epsilon^*_{h'}\cdot q)(\epsilon_h\cdot q)}{2M_W^2}(p+p')^\mu\ F_3(Q^2). \tag{14}$$

In a manifestly covariant calculation we can express F_i in terms of invariant integrals, but in LFD we can only determine them by taking linear combinations of the matrix elements $G^\mu_{h'h}$. We concentrate on F_2, as this form factor turns out to be particularly troublesome and find:

$$F_2^{+0} = \frac{1}{p^+}\left[-G^+_{++} + \frac{1}{\sqrt{2\eta}}G^+_{+0}\right], \quad F_2^{00} = \frac{1}{4\eta p^+}\left[(1-2\eta)G^+_{++} + G^+_{+-} - G^+_{00}\right], \tag{15}$$

where $\eta = Q^2/2M_W^2$. As the form factors are invariants, F_2^{+0} and F_2^{00} must be identical.

In what follows we consider in the manifestly covariant case two regularizations, DR_4, the usual dimensional regularization, and PV, Pauli–Villars regularization involving the struck fermion. In the light-front calculation we cannot use DR_4, but we can use dimensional regularization in the perpendicular momenta. Moreover, Pauli–Villars regularization is also possible in this case. The results for the combination $F_2 + 2F_1$, the anomalous magnetic moment are

$$(F_2^{+0} + 2F_1)^{\mathrm{DR}_2} = (F_2 + 2F_1)^{\mathrm{DR}_4} + \frac{g^2 Q_f}{4\pi^2}\frac{1}{6}, \\ (F_2^{00} + 2F_1)^{\mathrm{DR}_2} = (F_2 + 2F_1)^{\mathrm{DR}_4} + -\frac{g^2 Q_f}{4\pi^2}\left(\frac{1}{2\eta}\right)\left(\frac{1}{3} + \frac{2\eta}{9}\right). \tag{16}$$

Thus, we find that the vector anomaly in LFD breaks the Lorentz symmetry, i.e., $F_2^{+0} \neq F_2^{00}$ if DR_2 is used.

Using PV we find

$$(F_2^{+0} + 2F_1)^{\mathrm{PV}} - (F_2 + 2F_1)^{\mathrm{DR}_4} = \frac{2}{3}\,\frac{g^2 Q_f}{4\pi^2}.$$

$$(F_2^{00} + 2F_1)^{\mathrm{PV}} - (F_2 + 2F_1)^{\mathrm{DR}_4} = \frac{2}{3}\,\frac{g^2 Q_f}{4\pi^2}. \quad (17)$$

The PV results in LFD are identical to the PV result from the manifestly covariant calculation:

$$(F_2 + 2F_1)^{\mathrm{PV}}_{\mathrm{cov}} - (F_2 + 2F_1)^{\mathrm{DR}_4} = \frac{2}{3}\,\frac{g^2 Q_f}{4\pi^2}, \quad (18)$$

so that

$$(F_2^{+0} + 2F_1)^{\mathrm{PV}} = (F_2^{00} + 2F_1)^{\mathrm{PV}} = (F_2 + 2F_1)^{\mathrm{PV}}_{\mathrm{cov}}. \quad (19)$$

Thus, the PV results are absolutely convergent and restore completely the Lorentz symmetry. However, the fermion-mass-independent difference between the PV results and the manifestly covariant DR_4 result persists.

Fortunately, in the Standard Model, there exists a symmetry, namely in all three generations the sum of the charges of the fermions vanishes $\sum_f Q_f = 0$. This *anomaly-free* condition removes the discrepancies we found.

5 Summary

LF type I singularities can be handled, but form a serious hindrance to LF dynamics.

LF type II singularities cancel once the LF amplitudes corresponding to one and the same covariant diagram are added. If some amplitudes are dropped for 'physical reasons', the singularities persist.

LF Anomalies may be removed by a symmetry. This is the case in the Standard Model. If such a symmetry does not occur, a genuine problem arises. The requirement that a theory must be anomaly free may be applied as a bottom-up test of models.

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